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The wear contact problem with partial slippage☆

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Abstract

A method of analysing the variation in the stress state and shape of wearable bodies, subjected to the action of an oscillating shear load with partial slippage of the surface in the contact area is proposed. The method is based on the introduction of two time scales: the time of a single cycle of variation in the shear load and the time corresponding to a specified number of cycles. To estimate the shape variation of the surface due to wear a linear relation between the wear rate and the contact pressure and velocity of relative slippage is used. Cases of complete or partial removal of the wear particles from the friction zone are considered. As an example, the kinetics of the variation in the stress state in the contact of an elastic indentor, having a flat base and rounded edges, with an elastic half-space of the same material, is investigated. Analytical expressions for calculating the asymptotic values of the stresses and shape of the worn surface are obtained.

Partial slippage is characterized by the presence of stick and slip zones in the contact area. This type of friction occurs if the shear force applied to the interacting bodies does not exceed the limiting friction force in modulus. Recurrent slippage, which occurs when there is an oscillating shear force, causes wear of the interacting surfaces (fretting).[1] The variation in the shape of the surface as a result of wear leads to a redistribution of the contact stresses, which, in turn, affects the wear rate.

A numerical analysis of the evolution of the contact pressures at the interface of a cylindrical punch and an elastic half-plane during fretting, assuming that all the wear particles are removed from the friction zone,[2] indicates that the contact pressures differ considerably from the initial Hertz diagram after a certain number of cycles of variation in the shear load. An analytical method of investigating the evolution of the stresses during fretting while retaining the main assumptions made in the initial formulation of the problem[2] was proposed in Ref. 3. It was proved that an asymptotic solution of the problem exists and an analytical expression for it was obtained.

In this paper, we construct a solution of the wear contact problem with partial slippage in a more general formulation, allowing for the possibility of partial removal of the wear particles from the friction zone and the formation from the remaining particles of a surface layer possessing mechanical characteristics that differ from those of the main material. This formulation of the problem is confirmed by the actual conditions of contact interaction during fretting, observed in experiments.[1]

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1. Formulation of the problem

We will consider the plane contact problem of the interaction of two elastic bodies of the same materials, the shape of the surface of which in the undeformed state in a system of coordinates Oxz is described by monotonically non-decreasing functions $z = f_1(x)$ and $z = -f_2(x)$, where $f_i(x) = f_i(-x)$, $i = 1, 2$.

The body is acted upon by a normal load P and an oscillating shear force $Q(t)$, satisfying the condition

$$-Q^* \le Q(t) \le Q^*, \quad Q^* < \mu P \tag{1.1}$$

It is assumed that the forces are applied in such a way that the moment about the origin of coordinates is equal to zero and the punch does not rotate about the axis of symmetry, while, in the contact area at an arbitrary instant of time t, by virtue of condition (1.1) a central stick zone occurs, surrounded by two slip zones. The correctness of this assumption has been proved for bodies of cylindrical shape made of the same materials[4,5] and for an indentor with a flat base and rounded edges.[6]

The surfaces suffer wear in the slip zones, which leads to a variation in the shape of the contacting surfaces, and also to the formation of a thin layer of the wear particles (a third body) on their surfaces. We assume that the thickness of this layer $h(x, t)$ is proportional to the depth of the damaged (worn) layer $w(x, t)$ at the point x considered at the given instant of time t, i.e. $h(x, t) = \gamma w(x, t)(0 \le \gamma < 1)$, while its displacement u_{x3} in the direction of the Ox axis due to the action of the shear stress $q(x, t)$ is given by the expression

$$u_{x3} = \frac{h(x,t)}{k_t} q(x,t) = \frac{\gamma}{k_t} w(x,t) q(x,t) \tag{1.2}$$

The constant k_t represents the compliance of the layer in a tangential direction. The case $\gamma = 0$ corresponds to complete removal of the wear particles from the friction zone.

For different wear mechanisms we have the following relation between the linear rate of wear and the contact pressure $p(x, t)$ and the rate of slippage $V(x, t)$ (Archard's law[7])

$$\frac{\partial w}{\partial t} = K_w p(x,t) V(x,t) \tag{1.3}$$

where K_w is the wear coefficient, which depends on the properties of the interacting bodies, the temperature, etc.

In the problem in question the rate of slippage is given by the relation $V(x, t) = |\partial s(x, t)/\partial t|$, where the relative slippage $s(x, t)$ at the point x at the instant of time t is as follows:

$$s(x,t) = \begin{cases} 0, & |x| < c(t) \\ (u_{x1} - u_{x2} - u_{x3}) - \delta_x, & c(t) \le |x| \le a(t) \end{cases} \tag{1.4}$$

Here u_{x1} and u_{x2} are the tangential displacements of the contacting surfaces at the point x at the instant of time t, δ_x is the relative displacement of the contacting bodies in a tangential direction, $a(t)$ is the half-width of the contact area and $c(t)$ is the half-width of the stick zone on it.

Hence, the thickness of the damaged (worn) layer depends on the rate of relative displacement of the surfaces in the slip zones and is given by the relation

$$\frac{\partial w}{\partial t} = K_w p(x,t) \left| \frac{\partial s(x,t)}{\partial t} \right| \tag{1.5}$$

The change in the shape of the surface during wear leads to a redistribution of the contact stresses. Note that, in a single cycle of variation of the shear force $Q(t)$, the shear stresses change considerably, as well as the relative slippage of the surfaces and the size of the slip zone; at the same time, the contact pressures and size of the contact area change only slightly. In the case of contacting bodies made of the same materials, as is well known, the shear stress distribution generally has no effect on the contact pressures and the size of the contact area.[8] Considerable changes in the latter quantities occur when the number of cycles of oscillation of the shear force is increased, leading to a considerable change in the shape of the surface, and hence in this paper the contact characteristics will be analysed on two scale levels, connected with the time of a single cycle and with a number of cycles of oscillation of the shear force.

2. Increment in wear after one cycle

We will denote the contact pressure by $p(x, N)$ and the half-width of the contact area in the N-th cycle by $a(N)$. The size of the slip zone $(c(t), a(N))$, where the surface wear occurs, changes during one cycle, since the boundary of the stick zone changes in the range $c^*(N) \le c(t) \le a(N)$ (see Refs 4,5), where $c^*(N)$ is the half-width of the stick zone in the N-th cycle, corresponding to the limiting shear load $Q = Q^*$. The value of $c(t)$ is determined by the value of the shear force $Q(t) < Q^*$ applied at this instant, while the distribution of the shear stresses q_N in the contact area depends on the half-width of the slip zone at this instant of time, i.e. $q_N = q_N(x, c(t))$.[4] The following equilibrium condition is then satisfied

$$Q(t) = \int_{-a(N)}^{a(N)} q_N(x, c(t))dx \tag{2.1}$$

The value of the relative slippage s_N in the N-th cycle at a certain point x, where $c(t) \le |x| \le a(N)$ at the instant t also depends on the function $c(t)$, i.e. $s_N = s_N(x, c(t))$. It then follows from relation (1.5) that

$$\frac{\partial w(x,t)}{\partial t} = K_w p(x,t)\left|\frac{\partial s(x,c)}{\partial c}\frac{dc}{dt}\right| \tag{2.2}$$

Integrating relation (2.2) over the time interval Δt, corresponding to a single cycle of variation of the tangential load, and using the mean-value theorem, we successively obtain

$$\begin{aligned}\Delta w(x,N) &= K_w \int_{(N-1)\Delta t}^{N\Delta t} p(x,t)\left|\frac{\partial s(x,c)}{\partial c}\frac{dc}{dt}\right|dt \approx \\ &\approx 2K_w\tilde{p}(x,N)\left|\int_{c^*(N)}^{x}\frac{\partial \tilde{s}_N(x,c)}{\partial c}dc\right| = 2K_w\tilde{p}(x,N)|\tilde{s}_N(x,c^*(N))| \approx \\ &\approx K_w[p(x,N-1)|s_{N-1}(x,c^*(N))| + p(x,N)|s_N(x,c^*(N))|], \quad |x| > c^*(N)\end{aligned} \tag{2.3}$$

where $\tilde{p}(x, N)$ and $\tilde{s}_N(x, c*(N))$ are the values of the pressure and the slippage at a certain intermediate point of the N-th cycle. The last relation is obtained using the condition $\tilde{s}_N(x, x) = 0$ and the assumption that, for a differently directed action of the force $Q(t)$, the wear after a cycle is equal to twice its value after a half-cycle. The function $w(x, N)$, defined by relation (2.3), satisfies the condition

$$\Delta w(c^*(N), N) = \Delta w(a(N), N) = 0 \tag{2.4}$$

Conditions (2.4) are only written for positive values of x, since $w(x, N)$, $p(x, N)$ and $q(x, c)$ are even functions of the variable x.

Hence, in order to determine the increment of the wear after the N-th cycle, it is necessary to calculate the relative slippage $s_N(x, c^*(N))$ corresponding to the maximum value of the shear force Q^*.

3. Calculation of the function $s_N(x, c^*(N))$

The function $s_N(x, c^*(N))$ is found from relation (1.4) with $c(t) = c^*(N)$. To determine the tangential displacements u_{xi} of the boundary points of the indentor ($i = 1$) and of the base ($i = 2$), we will use the well-known relations between the contact stresses and the displacements of the boundary of an elastic half-plane[9]

$$\frac{du_{xi}}{dx} = -\frac{(1-2\nu_i)(1+\nu_i)}{E_i}p(x) - \frac{2(1-\nu_i^2)}{\pi E_i}\int_{-a}^{b}\frac{q(x')}{x-x'}dx' \tag{3.1}$$

where $p(x)$ and $q(x)$ are the normal and shear stresses on the boundary of the half-plane.

Taking into account the fact that the bodies in contact have the same constants of elasticity ($E_1 = E_2$ and ν_1 and ν_2), we obtain from relations (1.2), (1.4) and (3.1)

$$s(x, N) \equiv -s_N(x, c^*(N)) = s_N(c^*(N), c^*(N)) - s_N(x, c^*(N)) =$$

$$= \frac{\gamma}{k_t} w(x, N) q_N(x, c^*(N)) + \frac{2}{\pi E^*} \int_{-a(N)}^{a(N)} q_N(x', c^*(N)) \ln\left|\frac{x - x'}{c^*(N) - x'}\right| dx' \tag{3.2}$$

$$c^*(N) \le |x| \le a(N), \quad E^* = E/(2(1 - \nu^2))$$

The shear contact stress $q_N(x, c^*(N) \equiv q(x, N))$ in the N-th cycle, occurring in expression (3.2), corresponds to $Q(t) = Q^*$. The function $q(x, N)$ can be represented in the form

$$q(x, N) = \begin{cases} \mu p(x, N) - q^*(x, N), & |x| < c^*(N) \\ \mu p(x, N), & c^*(N) \le |x| \le a(N) \end{cases} \tag{3.3}$$

If the bodies in contact have the same constants of elasticity, the function $q^*(x, N)$ is found from the equation[6]

$$\int_{-c^*(N)}^{c^*(N)} \frac{q^*(t, N)dt}{x - t} = \frac{\pi \mu E^*}{2} H'(x, N), \quad |x| < c^*(N) \tag{3.4}$$

and the conditions

$$\int_{-c^*(N)}^{c^*(N)} q^*(x, N)dt = \mu P - Q^*, \quad q^*(c^*(N), N) = 0 \tag{3.5}$$

Here $H(x, N)$ is the gap between the surfaces in the unloaded state in the N-th cycle, while $H'(x, N)$ is the derivative of the function $H(x, N)$ with respect to x.

Since there is no wear inside the stick zone $|x| \le c^*(N)$ and correspondingly, no change in the shape of the surfaces, the function $H'(x, N)$ does not change as the number of cycles N increases, i.e. $H'(x, N) = f'(x)$ (here $f(x) = f_1(x) + f_2(x)$).

Problem (3.4), (3.5) has a unique solution, whence we can conclude that the function $q^*(x, N)$ and the half-width of the stick zone $c^*(N)$ do not change as the number of cycles increases, i.e.

$$q^*(x, N) = q^*(x, 0) = q^*(x), \quad c^*(N) = c^*$$

The solution of this system of equations can be written in the following form[8]

$$q^*(x) = \frac{\mu E^* \sqrt{c^{*2} - x^2}}{2\pi} \int_{-c^*}^{c^*} \frac{f'(t)dt}{(t - x)\sqrt{c^{*2} - t^2}}, \quad |x| \le c^* \tag{3.6}$$

The half-width of the stick zone c^* is found from the equation

$$-\frac{\mu E^*}{2} \int_{-c^*}^{c^*} \sqrt{\frac{c^* - t}{t + c^*}} f'(t)dt = \mu P - Q^* \tag{3.7}$$

4. Solution of the contact problem in the N-th cycle

The quantity $\Delta w(x, N)$ defines the increment of the linear wear of both bodies in the N-th cycle, and hence the total linear wear after N cycles can be calculated from the formula

$$w(x, N) = \sum_{n=1}^{N} \Delta w(x, n) \tag{4.1}$$

The quantity $w(x, N)$ represents the thickness of the layer, consisting of particles which have been separated from the main material, damaged after N cycles. Assuming that a certain number of these particles form an intermediate layer of thickness $h(x, N) = \gamma w(x, N)(0 \leq \gamma < 1)$ on the surface in the slip zones, we will define the gap between the surfaces in the N-th cycle in the unloaded state as

$$H(x, N) = f(x) + (1-\gamma)w(x, N) \tag{4.2}$$

The contact condition in the N-th cycle takes the form

$$u_{z1}(x, N) + u_{z2}(x, N) + u_{z3}(x, N) = D(N) - H(x, N) \tag{4.3}$$

where $u_{z1}(x, N)$, $u_{z2}(x, N)$ and $u_{z3}(x, N)$ are the displacements at the point x along the normal to the boundary of the bodies in contact ($i=1, 2$) and of the third body ($i=3$) due to their deformation, and $D(N)$ is the approach of the bodies.

Assuming that the thickness of the damaged layer $w(x, N)$ is small and comparable with the elastic displacements u_{zi} ($i=1, 2$), the boundary conditions will be considered on the undeformed surface. The relation between the contact pressures $p(x, N)$ and the elastic displacements $u_{zi}(x, N)$ then has the form[9]

$$u_{zi}(x, N) = -\frac{2(1-\nu_i^2)}{\pi E_i}\int_{-a(N)}^{a(N)} \ln|x-x'|p(x', N)dx' + \text{const} \tag{4.4}$$

To determine the displacements in the third body we will use the Winkler model[4]

$$u_{z3}(x, N) = \frac{\gamma}{\kappa}w(x, N)p(x, N) \tag{4.5}$$

The constant κ represents the modulus of elasticity of the third body.

If $f_1(x)$ and $f_2(x)$ are continuous functions, the unknown half-width $a(N)$ of the contact area can be found from the condition

$$p(a(N), N) = 0 \tag{4.6}$$

The contact pressure $p(x, N)$ also satisfies the equilibrium equation

$$\int_{-a(N)}^{a(N)} p(x, N)dx = P \tag{4.7}$$

Substituting expressions (4.2), (4.4) and (4.5) into the contact condition (4.3), written for an arbitrary point $|x| \leq a(N)$ and for $x = a(N)$, and bearing relation (4.6) in mind, we obtain the integral equation

$$\begin{aligned}&\frac{2}{\pi E^*}\int_{-a(N)}^{a(N)} \ln\left|\frac{a(N)-x'}{x-x'}\right|p(x', N)dx' + \frac{\gamma w(x, N)}{\kappa}p(x, N) + \\ &+ (1-\gamma)w(x, N) = f(a(N)) - f(x), \quad |x| \leq a(N)\end{aligned} \tag{4.8}$$

Eqs. (4.6)–(4.8), together with the equation

$$w(x, N) = w(x, N-1) + K_w[p(x, N-1)s(x, N-1) + p(x, N)s(x, N)] \tag{4.9}$$

obtained from relations (2.3) and (4.1), can be used to determine the contact pressures $p(x, N)$ and the half-width of the contact area $a(N)$ in the N-th cycle. The functions $p(x, N-1)$, $w(x, N-1)$ and $s(x, N-1)$ are known from the solution of the contact problem at the previous step.

We can then determine the component of the stresses $\sigma_{xx}(x, N)$ when $z=0$ using Muskhelishvili's solution.[8] We have

$$\sigma_{xx}(x, N) = -p(x, N) + \frac{2}{\pi}\int_{-a(N)}^{a(N)} \frac{q(x', N)dx'}{x'-x}, \quad -\infty < x < +\infty \tag{4.10}$$

5. Asymptotic analysis

Since the surface wear function in the zone of relative slip is a monotonically increasing function, the pressure in this zone approaches zero. At the edge of the stick zone the functions $\Delta w'(x, N)$ and $H'(x, N)$ suffer a discontinuity, and hence as $N \to \infty$ the contact pressure tends to the following distribution[8]

$$p_\infty(x) = \begin{cases} \dfrac{E^*}{2\pi\sqrt{c^{*2}-x^2}} \displaystyle\int_{-c^*}^{c^*} \dfrac{\sqrt{c^{*2}-t^2} f'(t)dt}{t-x} + \dfrac{P}{\pi\sqrt{c^{*2}-x^2}}, & |x| < c^* \\ 0, \quad c^* < |x| \le a_\infty \end{cases} \tag{5.1}$$

This solution satisfies Eqs. (4.6)–(4.8) when $0 \le \gamma < 1$.

The asymptotic expression for the shear contact stresses $q_\infty(x)$ follows from the conditions $s'(x) = 0$ when $|x| \le c^*$ and $q_\infty(x) = \mu p_\infty(x) = 0$ when $c^* < |x| < a_\infty$, which, when the interacting bodies are made of the same materials, lead to the integral equation

$$\int_{-c^*}^{c^*} \frac{q_\infty(t)dt}{x-t} = 0, \quad |x| < c^* \tag{5.2}$$

The solution of this equation, which satisfies the equilibrium condition

$$\int_{-c^*}^{c^*} q_\infty(t)dt = Q^* \tag{5.3}$$

has the form

$$q_\infty(x) = \begin{cases} \dfrac{Q^*}{\pi\sqrt{c^{*2}-x^2}}, & |x| < c^*, \\ 0, \quad c^* < |x| \le a_\infty \end{cases} \tag{5.4}$$

The half-width c^* of the stick zone in relations (5.1) and (5.4) is found from the solution of Eq. (3.7).

Note that the asymptotic value of the auxiliary function $q^*_\infty(x)$ is given by

$$q^*_\infty(x) = \mu p_\infty(x) - q_\infty(x) = \frac{\mu E^*}{2\pi\sqrt{c^{*2}-x^2}} \int_{-c^*}^{c^*} \frac{\sqrt{c^{*2}-t^2} f'(t)}{t-x} dt + \frac{\mu P - Q^*}{\pi\sqrt{c^{*2}-x^2}}, \quad |x| < c^* \tag{5.5}$$

This expression is identical with the function $q^*(x)$ (see (3.6)) and satisfies the conditions

$$q^*_\infty(x) = q^*(x) > 0, \quad |x| < c^*; \quad q^*_\infty(-c^*) = q^*_\infty(c^*) = 0$$

From relations (4.10), (5.1) and (5.4) we obtain

$$\sigma^\infty_{xx}(x) = \begin{cases} -p_\infty(x), & |x| < c_* \\ -\dfrac{2Q^* \operatorname{sign} x}{\pi\sqrt{x^2-c^{*2}}}, & |x| > c^* \end{cases} \tag{5.6}$$

Analysis of expressions (5.1), (5.4) and (5.6) shows that the contact stresses approach infinity at the ends of the stick zone as $N \to \infty$, and when $z = 0$ the stresses $\sigma^\infty_{xx}(x)$ have a discontinuity when $x = \pm c^*$.

If we take the limit as $N \to \infty$ in Eq. (4.8), we obtain the asymptotic expression for the surface wear when $0 \le \gamma < 1$

$$w_\infty(x) = \begin{cases} 0, \quad |x| < c^* \\ \dfrac{1}{1-\gamma}\left[f(a_\infty) - f(x) - \dfrac{2}{\pi E^*} \displaystyle\int_{-c^*}^{c^*} p_\infty(x') \ln\left|\dfrac{a_\infty - x'}{x - x'}\right| dx' \right], & c^* < |x| \le a_\infty \end{cases} \tag{5.7}$$

The asymptotic value of the half-width of the contact area is found from the relation $w_\infty(c*) = 0$, i.e. from the condition for the expression in brackets in Eq. (5.7) to be equal to zero when $x = c^*$. It follows from an asymptotic analysis that the asymptotic values of the stresses, and also the amount of wear particles removed from the friction zone, are independent of the properties of the third body. The properties of the latter only affect the wear rate and the overall quantity of wear particles, i.e. the quantity

$$W_\infty = 2\int_{c^*}^{a_\infty} w_\infty(x)dx$$

The function $w_\infty(x)$ is given by relation (5.7).

6. The algorithm for the numerical solution of the problem

We will introduce the following dimensionless parameters and functions

$$\begin{aligned}
&\bar{p}(\bar{x}, N) = \frac{2}{\pi E^*}p(\bar{x}R, N), \quad \bar{q}^*(\bar{x}) = \frac{2}{\pi\mu E^*}q^*(\bar{x}R), \quad \bar{K}_w = \frac{\pi\mu E^*}{2}K_w, \quad \bar{P} = \frac{2P}{\pi RE^*}\\
&\bar{Q} = \frac{2}{\pi\mu RE^*}(\mu P - Q^*), \quad \bar{x} = \frac{x}{R}, \quad \bar{a} = \frac{a}{R}, \quad \bar{c} = \frac{c}{R}, \quad \bar{\kappa} = \frac{\pi E^*}{2\kappa}, \quad \bar{k}_t = \frac{\pi\mu E^*}{2k_t}\\
&\bar{H}(\bar{x}, N) = \frac{H(\bar{x}R, N)}{R}, \quad \bar{s}(\bar{x}, N) = \frac{s(\bar{x}, N)}{R}, \quad \bar{f}(\bar{x}) = \frac{f(\bar{x}R)}{R}, \quad \bar{w}(\bar{x}, N) = \frac{w(\bar{x}R, N)}{R}
\end{aligned} \tag{6.1}$$

where R is a certain characteristic dimension, determined by the geometry of the touching surfaces.

Since $p(x, N)$ is an even function, the following relation holds

$$\int_{-a(N)}^{a(N)} \ln\left|\frac{x-x'}{a(N)-x'}\right| p(x', N)dx' = \int_0^{a(N)} \ln\left|\frac{x^2-x'^2}{a^2(N)-x'^2}\right| p(x', N)dx'$$

by using which we can convert the system consisting of Eqs. (4.6)–(4.8) to the following system of equations, written in dimensionless form

$$\begin{aligned}
&\int_0^{\bar{a}(N)} \ln\left|\frac{\bar{a}^2(N)-x'^2}{\bar{x}^2-x'^2}\right| \bar{p}(x', N)dx' + \bar{\kappa}\gamma\bar{w}(\bar{x}, N)\bar{p}(\bar{x}, N) +\\
&+(1-\gamma)\bar{w}(\bar{x}, N) = \bar{f}(\bar{a}(N)) - \bar{f}(\bar{x}), \quad |\bar{x}| \le \bar{a}(N)\\
&\int_0^{\bar{a}(N)} \bar{p}(\bar{x}, N)d\bar{x} = \frac{\bar{P}}{2}, \quad \bar{p}(\bar{a}(N), N) = 0
\end{aligned} \tag{6.2}$$

where

$$\begin{aligned}
&\bar{w}(\bar{x}, N) = \bar{K}_w[|s(\bar{x}, N)|\bar{p}(\bar{x}, N) + |\bar{s}(\bar{x}, N-1)|\bar{p}(\bar{x}, N-1)] + \bar{w}(\bar{x}, N-1)\\
&\bar{w}(\bar{x}, 0) = 0
\end{aligned} \tag{6.3}$$

$$\bar{s}(\bar{x}, N) = \begin{cases} 0, & 0 \le \bar{x} < \bar{c}^* \\ \displaystyle\int_0^{\bar{a}(N)} \ln\left|\frac{\bar{x}^2-x'^2}{\bar{c}^{*2}-x'^2}\right| \bar{q}(x', N)dx' + \gamma\bar{k}_t\bar{p}(\bar{x}, N)\bar{w}(\bar{x}, N), & \bar{c}^* \le \bar{x} \le \bar{a}(N) \end{cases} \tag{6.4}$$

$$\bar{q}(\bar{x}, N) = \begin{cases} \bar{p}(\bar{x}, N) - \bar{q}^*(\bar{x}), & 0 \le \bar{x} < \bar{c}^* \\ \bar{p}(\bar{x}, N), & \bar{c}^* \le \bar{x} \le \bar{a}(N) \end{cases} \tag{6.5}$$

The function $\bar{q}*(\bar{x})$ and the parameter $\bar{c}*$ are found from relations (3.6) and (3.7). The parameters $\bar{K}_w$, $\bar{P}$, $\bar{Q}$, $\bar{\kappa}$ and γ are given.

To calculate the evolution of the contact characteristics when wear occurs we used a step-by-step procedure. At each step the half-width of the contact area was divided into n sections $\{[t_j, t_{j+1}]\}_{j=1}^{n}$ (not necessarily equal) and the function $\bar{p}(\bar{x}, N)$ was approximated by a polygonal function $p(t)$ (with nodes at the points $\{t_j\}_{j=1}^{n+1}$) of the form

$$\begin{gathered} p(t) = p_j + \frac{p_{j+1} - p_j}{t_{j+1} - t_j}(t - t_j), \quad t_j \le t \le t_{j+1}, \quad j = 1, 2, \ldots, n \\ 0 = t_1 < t_2 < \ldots < t_n < t_{n+1} = a, \quad p_j = p(t_j), \quad p_{n+1} = p(a) = 0 \end{gathered} \tag{6.6}$$

As a result of substituting expression (6.6) into system (6.2) and integrating the linear function in each section, system (6.2) was reduced to a system of linear equations. The half-width of the contact area in the N-th cycle was found from system (6.2) by successive approximations. The functions $\bar{s}(\bar{x}, N-1)$, $\bar{w}(\bar{x}, N-1)$ and $\bar{p}(\bar{x}, N-1)$ were assumed to be known from the $(N-1)$-th step. The solution of system (6.2) enables us, in each N-th cycle, to determine, in dimensionless form, the contact pressure $\bar{p}(\bar{x}, N)$ and the shear stresses $\bar{q}(\bar{x}, N)$, the wear $\bar{w}(\bar{x}, N)$, the slip function $\bar{s}(\bar{x}, N)$ and the half-width of the contact area $\bar{a}(N)$.

7. An example of the calculation of the evolution of the contact characteristics during wear

We used the proposed algorithm to calculate the change in the shape of the bodies and the redistribution of the stresses when an elastic half-space interacts with an indentor in the form of a punch with a flat base and rounded edges (Fig. 1). In this case,

$$f(x) = \begin{cases} \dfrac{(x+d)^2}{2R}, & -a(N) \le x < -d \\ 0, & -d \le x < d \\ \dfrac{(x-d)^2}{2R}, & d \le x < a(N) \end{cases} \tag{7.1}$$

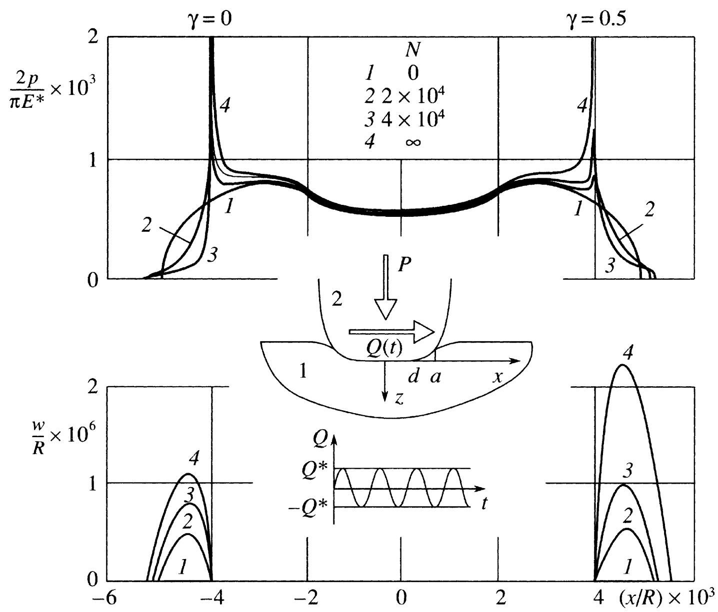

Fig. 1.

We took the following values of the dimensionless parameters for the numerical calculations

$$\bar{K}_w = 0.0495, \quad \bar{P} = 0.631\cdot 10^{-5}, \quad \bar{Q} = 0.788\cdot 10^{-6} \text{ or } \bar{Q} = 0.310\cdot 10^{-5}$$

$$\bar{d} = \frac{d}{R} = 0.002, \quad \bar{\kappa} = 1.1, \quad \bar{k}_t = 0.57$$

As follows from relations (6.1), for the chosen values of the dimensionless parameters the quantity $Q^*(\mu P)$ is equal to 0.875 or 0.509 respectively.

The initial pressure distribution function has the form[6]

$$p(x) = \pi(x, a) \triangleq \frac{E^*}{2\pi R}\left[2\sqrt{a^2 - x^2}\arccos\frac{d}{a} + (x+d)\ln\left|\frac{y_1 + y}{yy_1 + 1}\right| - (x-d)\ln\left|\frac{y_1 - y}{yy_1 - 1}\right|\right]$$

$$y = \operatorname{tg}\left(\frac{1}{2}\arcsin\frac{x}{a}\right), \quad y_1 = \operatorname{tg}\left(\frac{1}{2}\arcsin\frac{d}{a}\right)$$

The half-width of the contact area at the initial instant of time is found from the solution of the equation

$$a^2\arccos\frac{d}{a} - d\sqrt{a^2 - d^2} = \frac{2PR}{E^*}$$

From relations (3.6) and (3.7) we can determine the function $q^*(x)$ and the half-width c^* of the stick zone

$$q^*(x) = \mu\pi(x, c^*), \quad c^{*2}\arccos\frac{d}{c^*} - d\sqrt{c^{*2} - d^2} = \frac{2R(\mu p - Q^*)}{\mu E^*} \tag{7.2}$$

As was shown in Ref. 6, the boundaries of the stick zone $x = \pm c^*$ are always situated outside the flat part of the base $|x| \le d$, i.e. $c^* > d$.

During wear, as $N \to \infty$ the contact pressures tend to the asymptotic function $p_\infty(x)$, which, from relations (5.1) and (7.1), has the form

$$\begin{aligned} p_\infty(x) = {} & \frac{E^* c^*}{2\pi R}\frac{1+\eta^2}{1-\eta^2}\left\{\delta\sqrt{1-\delta^2} + \frac{2PR}{E^* c^*} + \left[1 - \frac{8\eta^2}{(1+\eta^2)^2}\right]\arccos\delta + \right. \\ & \left. + 2\eta\frac{1-\eta^2}{(1+\eta^2)^2}\ln\left|\frac{(\eta_1+\eta)(1-\eta\eta_1)}{(\eta_1-\eta)(1+\eta\eta_1)}\right| + \delta\frac{1-\eta^2}{1+\eta^2}\ln\left|\frac{\eta_1^2-\eta^2}{1-\eta^2\eta_1^2}\right|\right\}, \quad |x| < c^* \end{aligned} \tag{7.3}$$

$$\eta = \operatorname{tg}\left(\frac{1}{2}\arcsin\frac{x}{c^*}\right), \quad \eta_1 = \operatorname{tg}\left(\frac{1}{2}\arcsin\frac{d}{c^*}\right), \quad \delta = \frac{d}{c^*}$$

In Fig. 1 we show the results of a calculation of the evolution of the contact pressures $\bar{p}(\bar{x}, N)$ and of the function $\bar{w}(\bar{x}, N)$ when $\gamma = 0$ (on the left) and $\gamma = 0.5$ (on the right) for a number of cycles from $N = 0$ to $N \to \infty$ when $Q = 0.31\times 10^{-5}$. The pressure distribution undergoes considerable changes when there is fretting. The values of the pressures increase inside the stick zone and decrease in the slip zones, where surface wear occurs. When the number of cycles is increased without limit (i.e. when $N \to \infty$) the pressure on the boundary of the slip and stick zones (when $x = c^*$) increases without limit.

The distribution of the wear particles in the slip zones for the same values of the number of cycles N is shown in the lower part of Fig. 1. The maximum wear is in the middle parts of the slip zones. The absolute value of the right-hand derivative $\bar{w}'(\bar{x}, N)$ at the point c^* (and similarly the left-hand derivative $\bar{w}'(\bar{x}, N)$ at the point $-c^*$) increases as the number of cycles N increases. A comparative analysis of the evolution of the contact pressure and the contact wear function when the wear particles are completely removed from the friction zone ($\gamma = 0$) and partial removal with the formation of a third body ($\gamma = 0.5$) showed that the wear process in both cases can be divided into two stages: running-in and steady-state. The latter is characterized by distributions of the contact pressure and wear, represented analytically by expressions (5.1) and (5.7). The running-in process proceeds more rapidly when there is complete removal of the

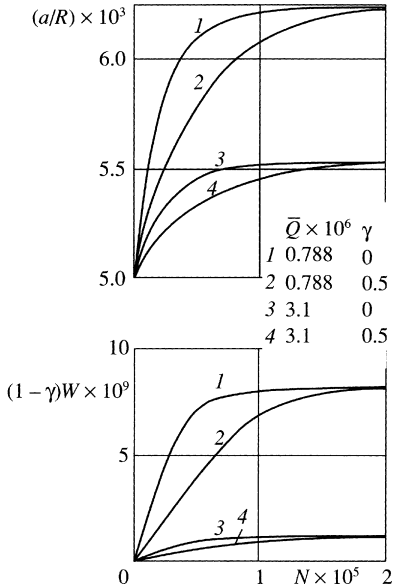

Fig. 2.

wear particles from the contact zone ($\gamma = 0$). When some of the wear particles accumulate in the friction zone, the volume of material undergoing destruction increases, i.e. the function $\bar{w}(\bar{x}, N)$ increases.

In Fig. 2 we illustrate the evolution of the half-width of the contact area $\bar{a}(N)$ during wear and the relation $(1-\gamma)W(N)$, where

$$W(N) = \int_{c^*}^{\bar{a}(N)} \bar{w}(\bar{x}, N)d\bar{x}$$

which characterizes the volume of the wear particles removed from the friction zone, as a function of the number of cycles N for different values of the shear forces applied to the indentor, in the case of complete removal of the wear particles from the friction zone ($\gamma = 0$) and incomplete removal ($\gamma = 0.5$). The dimensions of the contact area and the

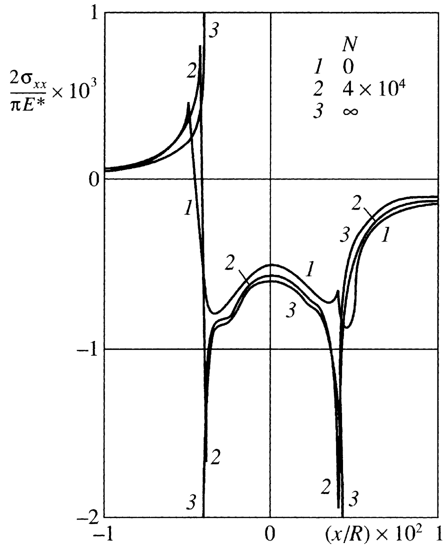

Fig. 3.

quantity $W(N)$ increase and approach certain constant values, which can be calculated analytically and depend on the quantity q (see formula (5.7)). The smaller the value of q and, correspondingly, the larger the value of Q^*, the greater the dimensions of the contact area and the greater the volume of the wear particles removed from the friction zone for the same value of the number of cycles N and their asymptotic values. The value of γ affects the running-in process: the larger the value of γ the greater the number of cycles required to arrive at a steady wear process.

The value of the wear coefficient K_w also affects the duration of the running-in process: an increase in its value reduces the duration of the running-in process. The solution of the system of equations obtained above also enables us to calculate the evolution of the gap between the surfaces, and also the stress state in the subsurface layers of material, which is important when analysing the nature of the fracture of the fretting surfaces. In Fig. 3 we show the distributions of the stresses σ_{xx} for $Q = 0.31 \times 10^{-5}$ for a different number of cycles N. At the beginning of the process the value of the tensile stresses for a chosen direction of action of the force Q^* is a maximum at the left end of the contact area ($x = -a(0)$). When the number of cycles is increased the point where the maximum tensile stresses are concentrated shifts to the boundary of the stick and slip zones, i.e. to the point $x = -c^*$. When an oscillating shear force $Q(t)$ acts, the presence of high amplitudes of the stresses at the ends of the stick zone leads to the damage accumulation at these points and to the nucleation of cracks, which have been detected in a number of experiments.[10]

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